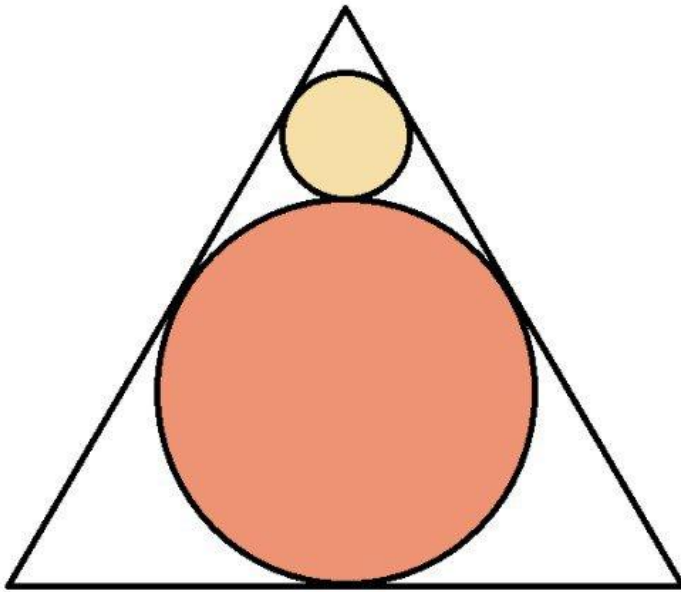
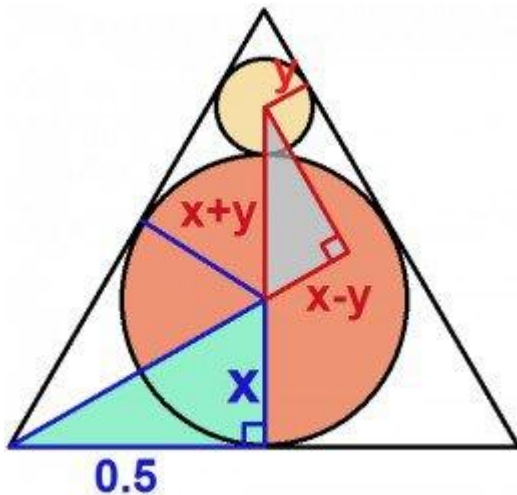


Problem 1



A circle is inscribed within an equilateral triangle. A smaller circle is inscribed in the space between the circle and two edges of the equilateral triangle. If the triangle has an edge length of 1, what are the radii of the large and small circles?



Solution: Call the radius of the larger circle x . Draw a 30° - 60° - 90° triangle whose three vertices are one corner of the outer triangle, the center of the larger circle, and the midpoint of the outer triangle's edge. The shortest side of this new triangle is x and the side of middle length is 0.5 . Using proportions of a 30° - 60° - 90° triangle, we can see that the ratio $x/0.5$ is equal to $1/\sqrt{3}$. This implies that $x = 0.5/\sqrt{3}$.

Now draw another 30° - 60° - 90° triangle whose hypotenuse spans the centers of the two circles. Call the radius of the smaller circle y . We can see that the longest side of this triangle has a length of $x+y$ and the shortest side has a length of $x-y$. Since the longest/shortest ratio equals $2/1$, we have

$$(x+y)/(x-y) = 2/1$$

$$x + y = 2x - 2y$$

$$x = 3y$$

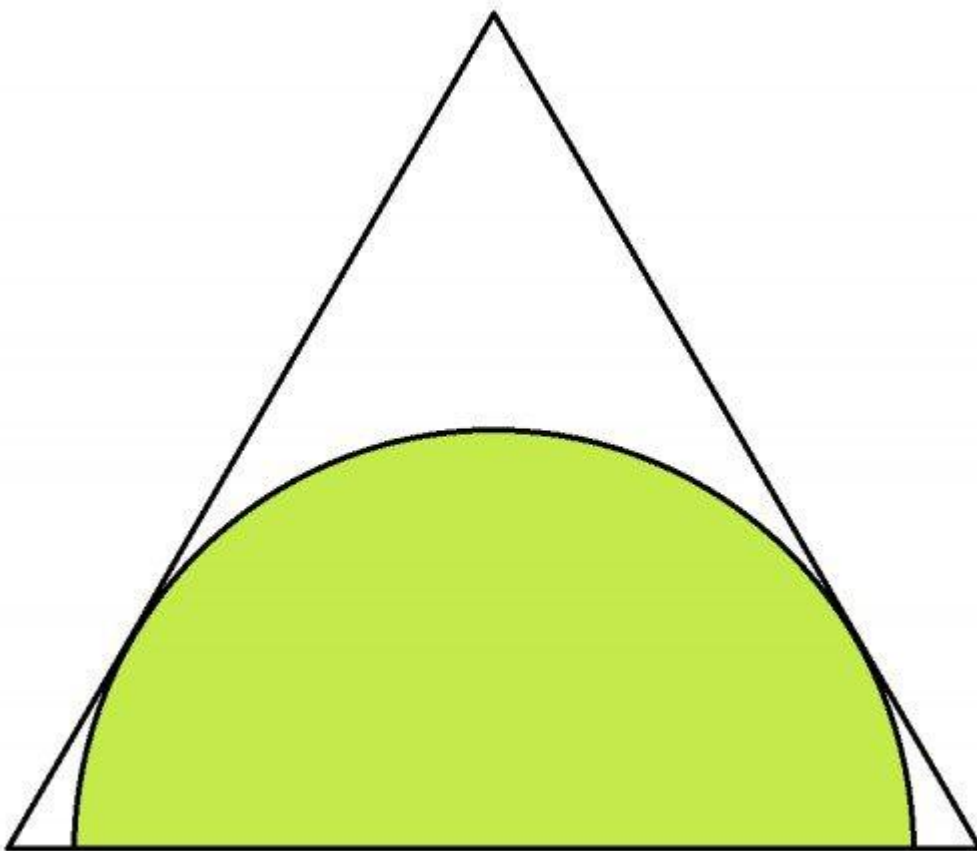
$$x/3 = y$$

$$(0.5/\sqrt{3})/3 = y$$

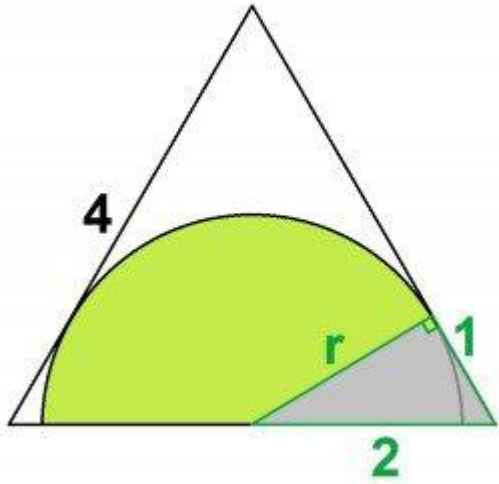
$$0.5/(3\sqrt{3}) = y$$

$$1/[6\sqrt{3}]$$

Problem 2



A half-circle is inscribed within an equilateral triangle such that the diameter of the half-circle is centered on one edge of the triangle and the arc is tangent to the other two sides. What is the diameter of the semi-circle if the triangle has an edge length of 4?

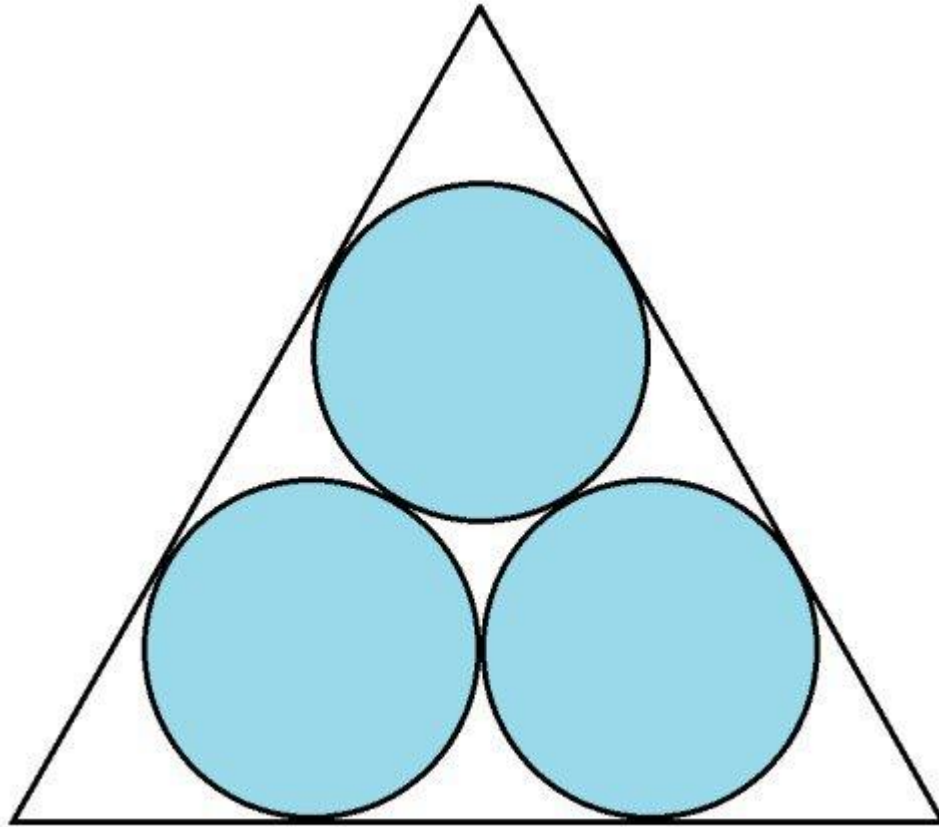


Solution: Call the radius of the semi-circle r . Draw a 30° - 60° - 90° whose vertices are one corner of the larger triangle, the midpoint of one side of the larger triangle, and the point of tangency between the semi-circle and the larger triangle. The hypotenuse of this 30° - 60° - 90° triangle has a length of 2. The longest leg as a length of r . Using the ratio

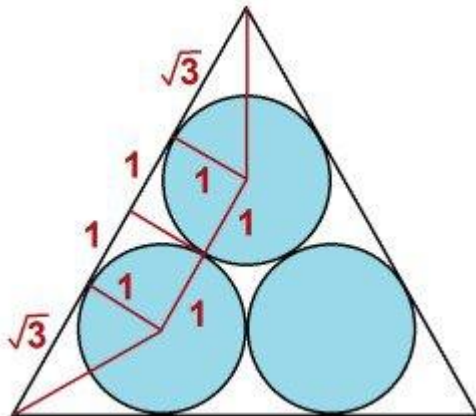
$$r/2 = \sqrt{3}/2$$

we can see that $r = \sqrt{3}$. Therefore the diameter of the semi-circle is $2\sqrt{3}$.

Problem 3



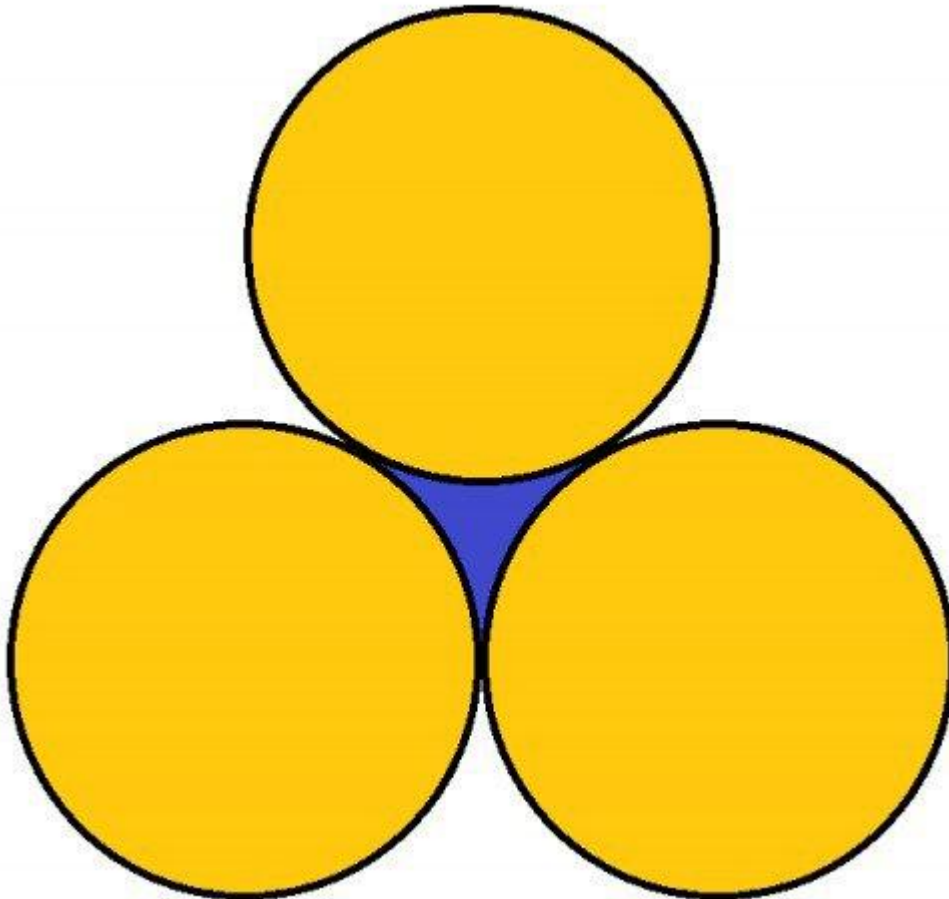
Three circles each with a radius of 1 are inscribed within an equilateral triangle such that the three circles are tangent to each other and to two edges of the triangle. What is the side length of the triangle?



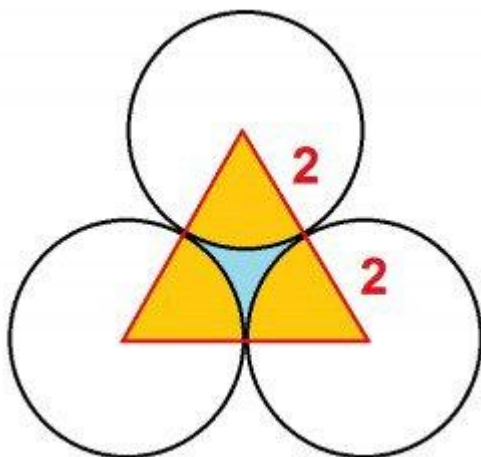
Solution: Pick two circles. Draw a line segment connecting their centers, and two line segments connecting the centers of the circles to the nearest corner of the equilateral triangle. Also draw line segments connecting the centers to the points of tangency along the edge of the triangle. As you can see using proportions of 30° - 60° - 90° triangles, the length of the equilateral triangle's edge is

$$\begin{aligned} & \sqrt{3} + 1 + 1 + \sqrt{3} \\ & = 2 + 2\sqrt{3} \end{aligned}$$

Problem 4



Three circles, each with a radius of 2, are mutually tangent. What is the area of the region bounded by the three circles, shown in blue above?



Solution: Draw an equilateral triangle that connects the centers of the circles. The side length of this equilateral triangle is 4. Using the area formula

$$\text{Area} = (s^2) \cdot \sqrt{3} / 4$$

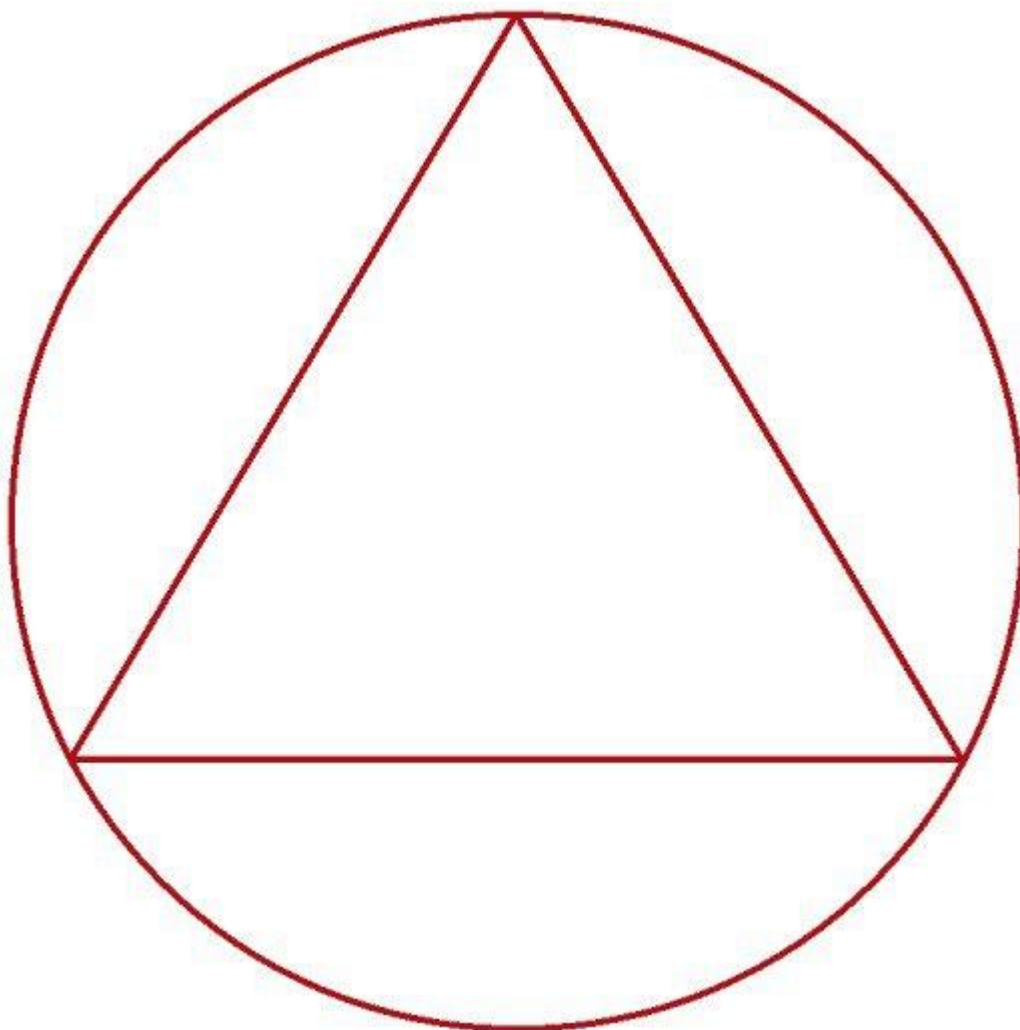
we can compute the area of this triangle:

$$\begin{aligned} \text{Area} &= (4^2) \cdot \sqrt{3} / 4 \\ &= 4 \cdot \sqrt{3} \end{aligned}$$

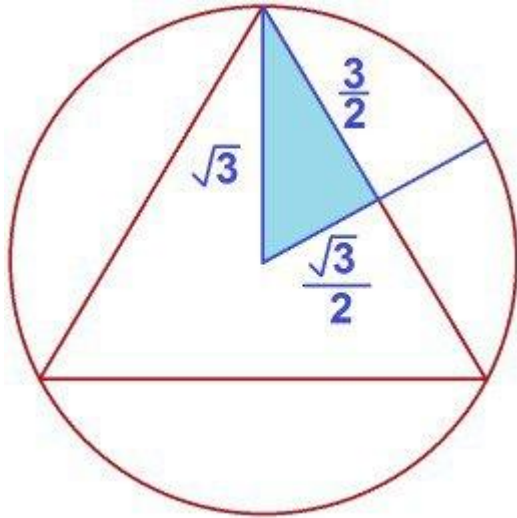
The three sectors shown in yellow have a combined area equal to that of a half-circle with a radius of 2. The combined area of these sectors is thus $(1/2)\pi \cdot 2^2 = 2\pi$. The area of the blue region is equal to the area of the triangle minus the area of these sectors. Thus the area of the blue region is

$$4 \cdot \sqrt{3} - 2\pi$$

Problem 5



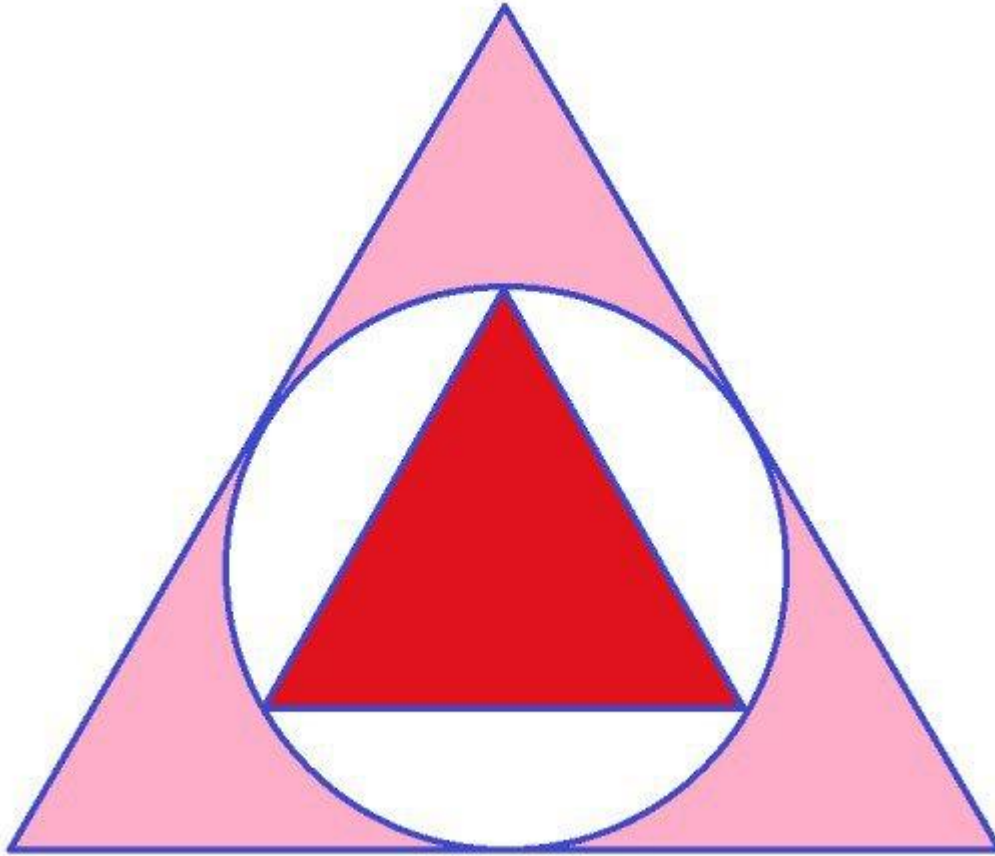
An equilateral triangle is inscribed within a circle whose radius is $\sqrt{3}$. What is the area of the triangle?



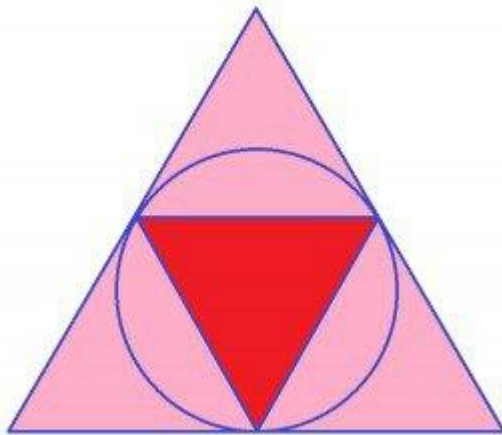
Solution: Draw a line from the vertex of the triangle to the center of the circle, and from the center of the circle to the midpoint of the triangle's edge. This forms a 30° - 60° - 90° whose side lengths are $\frac{\sqrt{3}}{2}$, $\frac{3}{2}$, and $\sqrt{3}$. The side length of the triangle is $2 \cdot \frac{3}{2} = 3$. Therefore, its area is

$$\begin{aligned} & \frac{(3^2) \cdot \sqrt{3}}{4} \\ &= \frac{9\sqrt{3}}{4} \end{aligned}$$

Problem 6

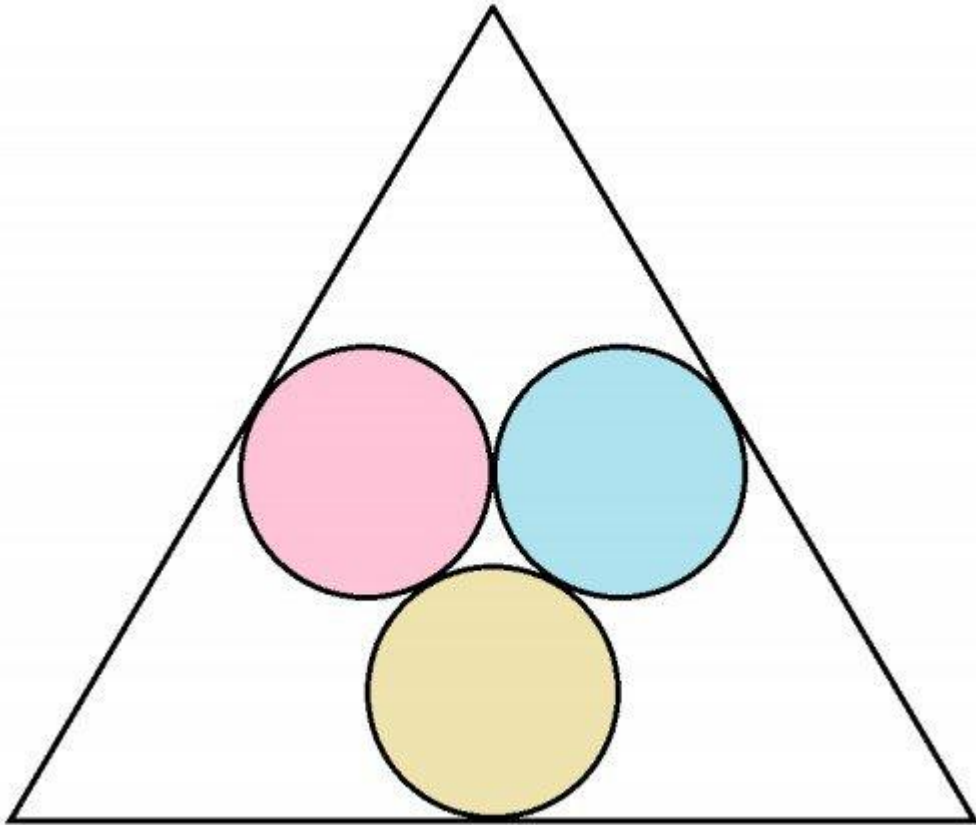


A circle is inscribed within an equilateral triangle. An equilateral triangle is inscribed within this circle. What is the ratio of the area of the larger triangle to the smaller triangle?

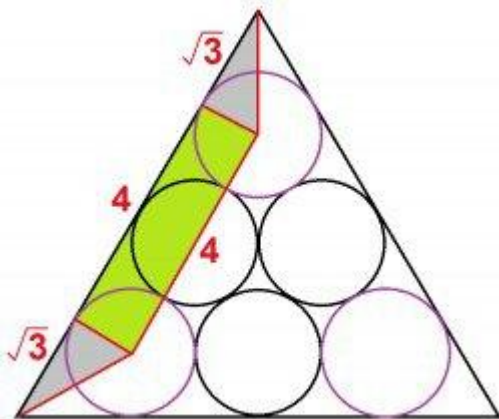


Solution: Rotate the circle so that the vertices of the smaller triangle lie on the midpoints of the edges of the larger triangle. That is, rotate the circle so that the smaller triangle is upside down. Now you can see that the smaller triangle takes up exactly $\frac{1}{4}$ the area of the larger triangle. Thus, the ratio of their areas is 4:1.

Problem 7



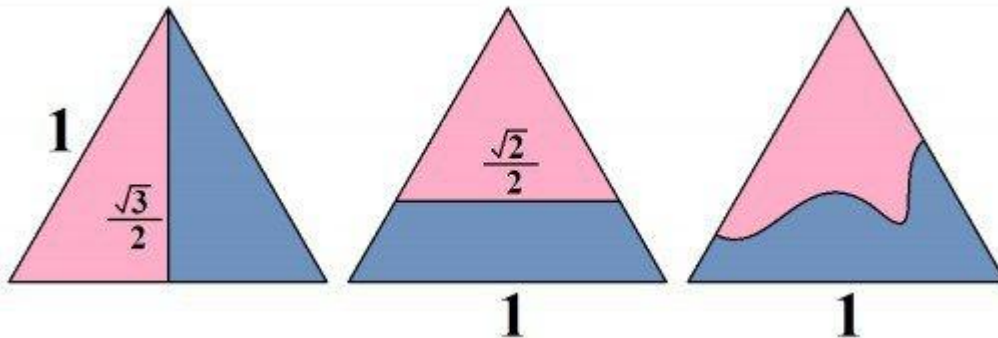
Three circles are inscribed within an equilateral triangle such that they are tangent to one another and to the midpoint of one edge of the triangle. If each circle has a radius of 1, what is the side length of the triangle?



Solution: Draw three more circles in the corners of the triangle so that they are tangent to the original three and to the edges of the triangle. These new circles also have a radius of 1. Draw the line segments as shown in the figure on the right so that you create a rectangle and two 30°-60°-90° triangles. The length of the edge of the triangle is

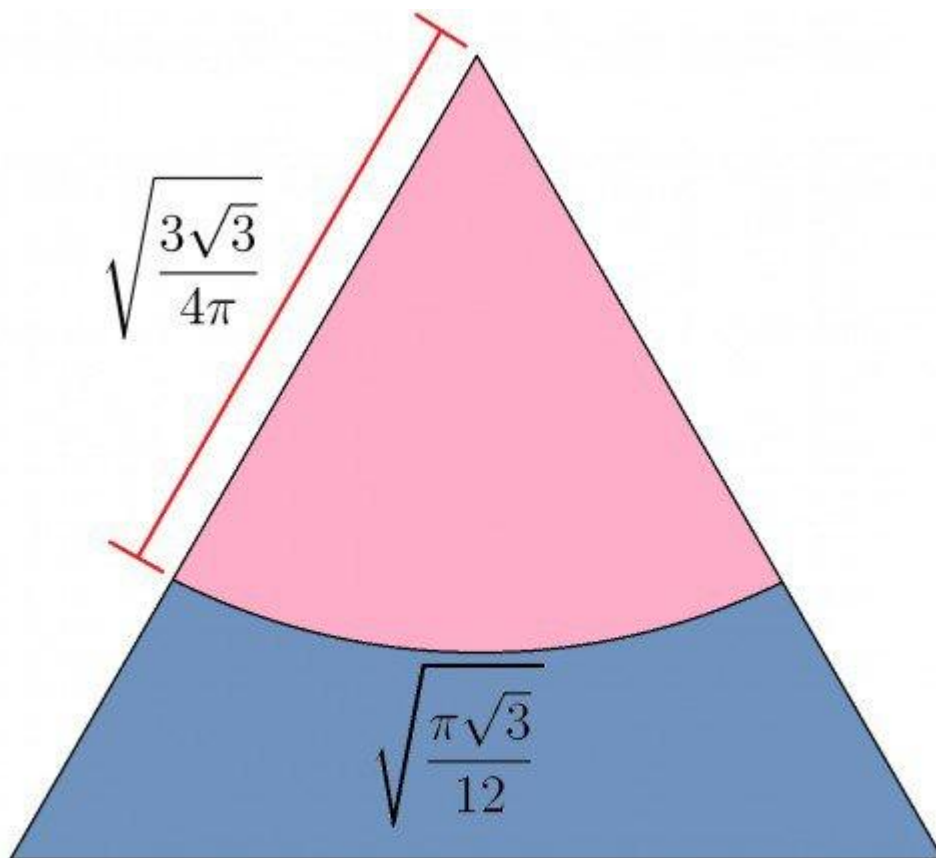
$$\begin{aligned} & \sqrt{3} + 4 + \sqrt{3} \\ & = 4 + 2\sqrt{3} \end{aligned}$$

BONUS Problem: Cut of Minimal Length That Splits Triangle into 2 Equal Pieces



Consider an equilateral triangle whose side length is 1. What is the shape and minimal length of a curve or line that will partition the triangle into two pieces of equal area? Some example lines and arcs are shown above.

Solution: The curve that cuts the triangle into two pieces and has the minimum length possible is a circular arc centered at one of the vertices of the triangle. The radius of this arc is $\sqrt{\frac{3\sqrt{3}}{4\pi}} \approx 0.643037$, and the total curved length of the arc is $\sqrt{\frac{\pi\sqrt{3}}{12}} \approx 0.673387$. See image below.



To obtain this value of r , we first note that half the area of the triangle is $\sqrt{3}/8$, and that a circular arc centered at the vertex will create a $1/6$ wedge. This means we solve the equation

$$\sqrt{3}/8 = (\pi r^2)/6$$

$$6\sqrt{3}/(8\pi) = r^2$$

$$\sqrt{3\sqrt{3}/(4\pi)} = r$$

The arc length is $2\pi r/6 = (\pi/3)r$, or $\sqrt{\pi\sqrt{3}/12} \approx 0.673387$.

In comparison, a straight line cut from one vertex to the midpoint of the opposite side has a length of $\sqrt{3}/2 \approx 0.866025$, and a straight line cut parallel to the base has a length of $\sqrt{2}/2 = 0.707107$.